Estimating Planetary Habitability: Theory of convex optimization and metaheuristics

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Problem

The possibility of life in a form known or unknown to us (habitability) is a hard problem to quantify. We propose a different metric, a Cobb Douglas Habitability Score (CDHS) based on the theory of convex optimization. The proposed metric, with exponents accounting for metric elasticity, is endowed with analytical properties that ensure global optima, and scales up to accommodate finitely many input parameters. The proposed work describes a decision-theoretical model using the power of convex optimization and algorithmic machine learning.

Basic Concepts

We have proposed the new model of the habitability score in using a convex optimization approach. In this model, the Cobb Douglas function is reformulated as Cobb-Douglas habitability production function (CD-HPF) to compute the habitability score of an exoplanet,

$$\mathbb{Y} = f\left(R, D, T_s, V_e\right) = K\left(R\right)^{\alpha} \cdot \left(D\right)^{\beta} \cdot \left(T_s\right)^{\gamma} \cdot \left(V_e\right)^{\delta}, \tag{1}$$

where the same planetary parameters are used – radius R, density D, surface temperature T_s, and escape velocity V_e. $\mathbb{Y}$ is the habitability score CDHS, and f is defined as CD-HPF [a]. The goal is to maximize the score, $\mathbb{Y}$, where the elasticity values of each parameter are subject to the condition $\alpha + \beta + \gamma + \delta < 1$.

[a] Elasticities K, α, β, γ and δ need to be estimated

Theorem

Theorem: If global maxima for CDHS, i.e.

$$\log(Y) = \frac{1}{1 - \sum\limits_{i=1}^{n} \alpha_i} \log \left\{ k \prod_{i=1}^{n} \left(\frac{x_i p}{w_i} \right)^{\alpha_i} \right\} \tag{2}$$

holds, then the same condition for the global maxima will continue to hold if an additional input parameter is inserted in the habitability function CD-HPF, i.e., if

$$\log(Y_{\text{new}}) = \frac{1}{1 - \sum\limits_{i=1}^{n+1} \alpha_i} \log \left\{ k \prod_{i=1}^{n+1} \left(\frac{x_i p}{w_i} \right)^{\alpha_i} \right\}$$

holds as well. Further, it follows that the elasticity condition for DRS for $n+1$ parameters is true, i.e. $1 - \sum\limits_{i=1}^{m+1} \alpha_i > 0$, if the elasticity condition for DRS for n parameters, i.e. $1 - \sum\limits_{i=1}^{m} \alpha_i > 0$, holds.[a]

[a] Remark: The above theorem validates the scalability of the model describing The habitability score CDHS under additional parameters different from R, D, T_s and V_e ensuring global maxima for habitability.

References

[1] Saha et.al.: *Theoretical Validation of Potential Habitability via Analytical and Boosted Tree Methods: An Optimistic Study on Recently Discovered Exoplanets*, Astronomy and Computing (Elsevier), 2018.

[2] Bora et. al. *CD-HPF: New Habitability Score Via Data Analytic Modeling*, Astronomy and Computing (Elsevier), 2016.

Mitigating oscillations about the optima

We have implemented a more efficient algorithm to compute the elasticity values. We devised a sensitive method which would mitigate oscillatory nature of Newton-like methods around the local minima/maxima. Although theoretically sound, algorithmic implementations of most of these methods face convergence issues in real time due to the oscillatory nature. Stochastic Gradient Descent was used to find the minimal value of a multivariate function, when the input parameters are known. Output elasticity (α, β, γ or δ) of Cobb-Douglas habitability function is the accentual change in the output in response to a change in the levels any of the inputs. Accuracy in elasticity values is crucial in deciding the right combination for the optimal CDHS, where different approaches are analyzed before arriving at final decision. In the next subsections, we show how the elasticities were computed on the example of α and β. Once they are computed, we repeat the procedure to compute other elasticities, γ and δ. Output elasticity (α, β, γ or δ) of Cobb-Douglas habitability function is the accentual change in the output in response to a change in the levels any of the inputs. Accuracy in elasticity values is crucial in deciding the right combination for the optimal CDHS, where different approaches are analyzed before arriving at final decision. Given a scalar function $F(x)$, stochastic gradient ascent finds the $\max_x F(x)$ by following the slope of the function, albeit in random directions. This algorithm selects initial values for the parameter x and iterates to find the new values of x which maximizes $F(x)$ (CDHS). Maximum of a function $F(x)$ is computed by iterating through the following step,

$$x_{n+1} \leftarrow x_n + \chi \frac{\partial F}{\partial x}, \tag{3}$$

where x_n is an initial value of x, x_{n+1} the new value of x, $\frac{\partial F}{\partial x}$ is the slope of function $Y = F(x)$ and χ denotes the step size, a random variable typically which is greater than 0 and forces the algorithm to make jumps in the direction of the new update. Since our optimization problem is constrained, SGA although accelerates convergence to optimal elasticity values, could be further improved. Standard Swarm optimization methods are not suitable for constrained optimization problems such as the one in question. The standard PSO algorithm does not ensure that the initial solutions are feasible, and neither does it guarantee that the individual solutions will converge to a feasible global solution. Solving the initialization problem is straightforward, re-sample each random solution from the uniform distribution until every initial solution is feasible. To solve the convergence problem each particle uses another particle's p_{best} value, called l_{best}, instead of its own to update its velocity. We represent non-strict inequality constraints when optimizing using a particle swarm. Strict inequalities and equality constraints need to be converted to non-strict inequalities before being represented in the problem. Introducing an error threshold ϵ converts strict inequalities of the form $g_k{}'(x) < 0$ to non-strict inequalities of the form $g_k(x) = g_k{}'(x) + \epsilon \leq 0$. A tolerance τ is used to transform equality constraints to a pair of inequalities,

$$\begin{aligned} g_{(q+l)}(x) &= h_l(x) - \tau \leq 0, \qquad l = 1 \ldots r, \\ g_{(q+r+l)}(x) &= -h_l(x) - \tau \leq 0, \qquad l = 1 \ldots r. \end{aligned}$$

Thus, r equality constraints become $2r$ inequality constraints, raising the total number of constraints to $s = q + 2r$. For each solution p_i, c_i denotes the constraint vector where, $c_{ik} = \max\{g_k(p_i), 0\}$, $k = 1 \ldots s$. When $c_{ik} = 0$, $\forall k = 1 \ldots s$, the solution p_i lies within the feasible region. When $c_{ik} > 0$, the solution p_i violates the k-th constraint.

CDHS Metaheuristics

The representation of CDHS estimation under CRS as a CO problem is given by,

$$\begin{aligned} \underset{\alpha,\beta,\gamma,\delta}{\text{minimize:}} \quad & Y_i = -R^{\alpha}.D^{\beta}, \; Y_s = -V_e{}^{\gamma}.T_s{}^{\delta} \\ \text{subject to:} \quad & -\phi + \epsilon \leq 0, \quad \phi - 1 + \epsilon \leq 0, \quad \forall \phi \in \{\alpha, \beta, \gamma, \delta\} \\ & (\alpha + \beta - 1) - \tau \leq 0, \quad (\gamma + \delta - 1) - \tau \leq 0, \\ & (1 - \alpha - \beta) - \tau \leq 0, \quad (1 - \gamma - \delta) - \tau \leq 0. \end{aligned} \tag{4}$$

Under DRS, the last two constraints for Y_i and Y_s are replaced with, $\alpha + \beta + \epsilon - 1 \leq 0$ & $\gamma + \delta + \epsilon - 1 \leq 0$.